NASA Conference Publication 3354

National Educators' Workshop: Update 96

Standard Experiments in Engineering Materials Science and Technology

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Proceedings of a workshop sponsored jointly by
the United States Department of Energy, Los Alamos, New Mexico,
the National Aeronautics and Space Administration, Washington, D.C. ,
the Norfolk State University, Norfolk, Virginia, and
the National Institute of Standards and Technology, Gaithersburg, Maryland
and held in
Los Alamos, New Mexico
October 27-30, 1996

July 1997

RELAXATION AND RESISTANCE MEASUREMENT

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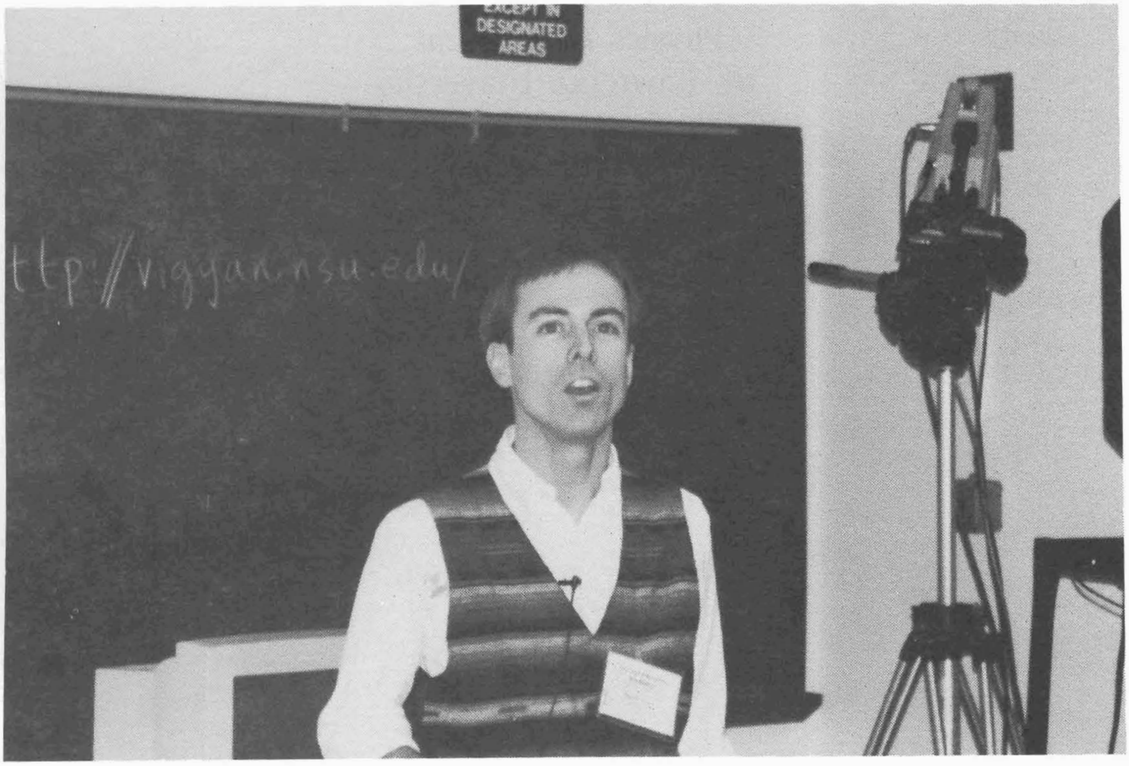

Relaxation and Resistance Measurements

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Key words: Resistance measurement, Laplace's equation, numerical analysis

Prerequisite knowledge: Elementary differential equations, elementary electricity and magnetism, experience with computer spreadsheeting.

Objective: To introduce four-point resistance measurement techniques and numerical methods for solving differential equations.

Equipment and Supplies:

1. Microcomputer
2. Computer spreadsheet program, e.g. *Microsoft Excel 4.0* or higher. (Any spreadsheet program which allows for iterative calculation of circular references in formulas should work.)
3. Brass shim stock (preferably 0.002″ or thinner)
4. Reversible DC current supply
5. Ammeter
6. Microvoltmeter
7. Soldering iron
8. Solder
9. Tinpunch or tinsnips

Introduction:

Electrical resistivity measurement is a cheap atomic microscope. This measurement allows one to 'see' the individual charge carriers through their interaction with the surrounding atoms. If one simultaneously measures resistivity and another electrical quantity, the Hall effect, one can calculate the density, mobility, and the sign of the charge carriers, all microscopic quantities.

Accurate measurement of low-resistivity materials requires eliminating the ohmmeter's lead wire resistance from the measurement. Four-wire techniques allow this elimination and are thus universally used for precision measurements. Using such a technique, wires are attached to the sample at four points as shown in Figure 1. A known current, I, flows between two of the wires, and the voltage difference, ΔV, across the other two wires is measured to obtain a resistance, R, as

$$R = \frac{\Delta V}{I} \tag{1}$$

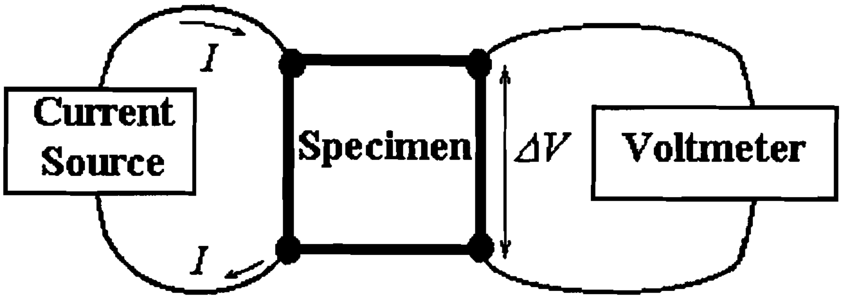

Figure 1. Schematic of a four-point resistance measurement. Current flows between the left-hand corners of the specimen. Voltage is measured across the right-hand corners.

The measured resistance will depend on the dimensions of the sample, and the placement of the wires. It will also be proportional to the resistivity, ρ, of the material, which is independent of these factors. The conversion between resistance and resistivity can be found either empirically or by calculation, if the specimen geometry is precisely known.

If the composition of the material being studied is not uniform, then we can think of the measured resistivity as the sum of the volume weighted local values of resistivity. It would be surprising if the resistivity of all volume elements were weighted equally. To determine how the local resistivities are averaged, we can ask what effect a small local perturbation will have on the measured resistance. This requires calculating the change in the voltage due to perturbing the local resistivity.

The measured voltage, ΔV, is simply the difference in the electrical potential, Φ, at the two points where the voltage leads touch the sample. The electrical potential in a material in which charge is free to flow obeys Laplace's equation

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{2}$$

subject to certain boundary conditions at the edge of the specimen. (∇^2 is called the Laplacian operator.) This is a specific case of a broad class of differential equations problems known as boundary value problems. These can sometimes be solved exactly and analytically, but most often must be solved using numerical techniques if an approximate solution is sufficient.

We can approximate a flat, square sample by a two-dimensional grid of points, as shown in Figure 2, and replace the Laplacian with the corresponding finite difference equation. If we use the symbol $\Phi_{i,j}$ to represent the potential at a point in row i and column j of such a grid, then equation *2* becomes[1]

$$\Phi_{i,j}=(\Phi_{i-1,j}+\Phi_{i+1,j}+\Phi_{i,j-1}+\Phi_{i,j+1})/4, \tag{3}$$

that is, the electrical potential at any point is just the average of the electrical potential values of its neighbors. While this finite difference equation is true in the interior of a sample of material, on the boundaries it is not. Since there is no current flow across the boundaries, except at the current probes, the derivative of the potential normal to the surface, $\partial\Phi/\partial n$ (with 'n' representing the direction normal to the boundary), is zero at the boundaries except at the current probes. If we consider a gridpoint at the top of

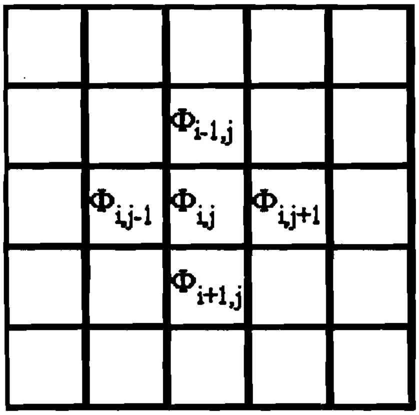

Figure 2: Schematic of finite-difference approximation of a boundary-value problem. Fij is the electric potential at a gridpoint in row i and column j.

the sample (i=1), then the fictitious row 0 would have the same potential as row 2, and so Equation 3 becomes

$$\Phi_{1j}=(2\Phi_{2,j}+\Phi_{1j-1}+\Phi_{1j+1})/4. \tag{4}$$

To calculate the electrical potential throughout the sample, we assume an initial value for it, and then use Eqs. 3 and 4 iteratively to calculate the potential at each point in the interior and at the boundary. Such a technique is called 'relaxation', and it generally converges to the correct potential. The easiest way to 'relax' is to program a computer to do all the work, using either a programming language or a spreadsheet to iterate Eqs. 3 and 4. (A spreadsheet is simpler to program and allows one to monitor the computer's progress.)

We will explore how numerical techniques like relaxation can be used to solve boundary-value problems like Laplace's equation by solving a practical question. Namely, we will investigate how a resistance measurement averages the local resistivities, by calculating the effects of a 'specimen hole' (a small region of infinite resistance) placed somewhere inside a conducting specimen. Next, we will move the specimen hole around within the specimen to see how its effects vary with position. We will then test our results experimentally.

Procedure I - Computational

In this procedure, we solve Laplace's equation numerically using the technique of relaxation. The student creates a spreadsheet which simulates a square resistance measurement specimen with current leads and voltage leads at its four corners. The student inserts a moveable specimen hole in the spreadsheet, and its effect on the measured resistance is calculated as the hole is moved.

1. Create a spreadsheet. In the first cell, *A1*, type '*Phi0*'. In cell *A2* enter the number 0.

2. Several columns to the right and at least four rows below the top of the spreadsheet, enter a formula for one cell which consists of the average of its four nearest neighbors. Copy and paste this formula onto a region of about 10×10 cells. This will be the interior of your sample, to be surrounded by a one-cell thick boundary on all sides.

3. If the computer displays an error message like 'Cannot resolve circular references', it is because you asked the computer to calculate a cell's value using its neighbor's value, which, in turn, requires it to know the original cell's value. In order to get the computer to solve these formulas iteratively, you need to find the *Calculations* menu. In *Microsoft Excel 5.0*, this is found by selecting the following menus, in order: *Tools*, *Options*, *Calculation*. (In *Excel 4.0*, the sequence is *Options*, *Calculations*.) Mark an 'X' in the *Iteration* box in the *Calculations* menu, and type 0 for *Maximum Change*. Depending on the speed of your machine, you may want to increase *Maximum Iterations* from 100 to 1000. Now the spreadsheet will iterate circular definitions automatically. You may want to change column widths to view the entire spreadsheet at once.

4. Your sample should be very boring, having zero potential throughout. You need to impose boundary conditions. Start at the empty row directly above your rectangle of zeroes. Enter the formula equivalent of Eq. 4 in one cell at this edge. Cut and paste this formula onto this entire top row of your spreadsheet, except for the top two corners.

5. Repeat step 4 for the left-hand, right-hand, and bottom edges. You should be able to figure out how to alter Eq. 4 to suit each edge.

6. Set the top left corner equal to *+Phi0* (the value in cell *A2*), and the bottom left corner to *-Phi0*. These are your current probes.

7. In each of the two righthand corners, type a formula to average its two nearest (nondiagonal) neighbors. These right-hand corners will be the voltage probes.

8. Change the value of *Phi0* from zero to 100. Watch the spreadsheet update. If values are still changing when the calculation stops, press the *F9* key to continue iterating, or increase *Maximum Iterations* in the *Calculations* menu. In cell *B2* of the spreadsheet, calculate the difference in potential between the two voltage probes. In cell *C2*, calculate the ratio of this value to *Phi0*. For our purposes, we can let this ratio stand in for the measured resistance. Record its value, and verify that it doesn't change as you change *Phi0*.

9. To analyze the effect of resistivity inhomogeneities, we will put a region of high resistivity -- a 'specimen hole' -- in the center of the spreadsheet. Find the cell closest to the center, and enter a zero value in this cell. (The zero just makes it easier to find the cell later.) Imagine that you have just cut this region out of the sample. Its four nearest nondiagonal neighbors are now boundary points. Change the formulas in these cells to reflect their new status (as in Step 5). Record the new resistance once the spreadsheet has been altered and has updated itself. (If any cells 'blow up', copy the formulas of each edge, corner, and the center onto some clear region of the spreadsheet, erase the offending region, and paste those formulas back. This may spare you the frustration of starting again from scratch.)

10. Calculate the relative change in the measured resistance due to having such a specimen hole...
    a) at the center of the sample
    b) near the center of the lefthand edge
    c) near the center of the righthand edge
    d) near the center of the top edge
    e) near the center of the bottom edge

Procedure II - Experimental

In this procedure, the student compares the predictions of the spreadsheet with actual resistance measurements made by punching holes in various locations of an actual square specimen.

1. Cut a square piece of brass shim stock, several inches wide. Solder wires to its four corners.

2. Connect two adjacent wires from your sample to a current supply with a digital ammeter in series. Attach the other two leads to a voltmeter. If there is a nonzero voltage when the current is zero, you will need to measure the voltage by either reversing the current or turning it on and off, and then calculating the resistance from

$$R = \frac{\Delta V_2 - \Delta V_1}{I_2 - I_1}, \tag{5}$$

   where the numerator is the difference between the voltages measured with either of the two currents, and the denominator is the difference in the currents.

3. You can increase the local resistivity of the material by punching or cutting a specimen hole in the brass. (Alternately, you can decrease the local resistivity by melting some solder onto a point somewhere on the brass.) Since we can't undo these changes, it is useful to note that for sufficiently small perturbations, the effect of two or more specimen holes is additive. Measure the effect of a local specimen hole for the same positions mentioned in step *10* above, and compare to your computational results.

Notes to the instructor:

1. This project was designed as a single-student project for a one-semester junior-level E&M course with no lab. Students had previously been assigned a two-dimensional Dirichlet problem (a boundary-value problem in which the potential on the boundary is known) to be done using a spreadsheet. This project could also either be pursued as a multi-week lab project or expanded into a senior research project.

2. One thing we didn't take into consideration was the effect of our 'specimen hole' in changing the current flow in the sample. Including this effect is cumbersome, and for specimen holes near the centers of the edges of the sample, it is not significant. (For specimen holes close to the current probes, it can be.) To verify this, check whether the effect calculated for a specimen hole near the top edge equals the effect for a specimen hole at the corresponding point near the bottom edge, or whether the effect for a specimen hole near the left edge equals the effect for a specimen hole near the right edge. By symmetry they should.

3. For steps *10.a-c* of the calculations procedure, increasing (decreasing) the local resistivity should increase (decrease) the measured resistance, while for steps *10.d-e*, any changes to the local resistivity will have the opposite effect.[2] If we think of the measured resistivity as a weighted average of local values of resistivity, then for these regions, there is negative weighting, as illustrated in Figure 2. This

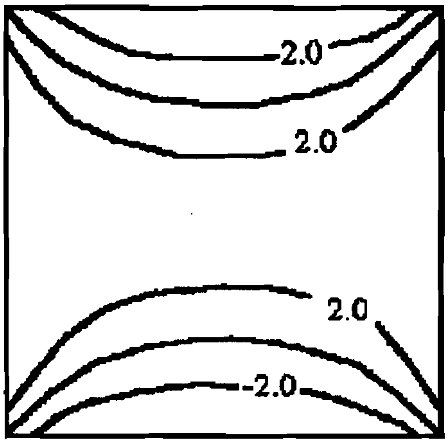

Figure 3: "Resistive weighting function" for specimen with two current probes on the left-hand corners and two voltage probes on the right-hand corners, after Ref. 2. Regions of negative weighting are at the top and bottom edges.

has interesting implications for the nature of resistivity measurements. As food for thought, you might ask your students to think of the implications of making their course grade a weighted average of homework, lab, and exams, in which the exams are negatively weighted. Would they want to do well or poorly on the exams?

4. As to *why* there is negative weighting, consider a model consisting of four resistors arranged as the four sides of a square.[2] Send current between two adjacent corners of this square and measure the voltage between the other two corners. The ratio of the voltage difference to the current can be calculated, and the effects of changing any of the four resistances can be predicted. The predictions of this simple model correspond to what is calculated and measured for the more complicated system above.

5. Despite its low resistance, brass works better for this demonstration than carbonized paper, for which the measured resistance fluctuates, and aluminized mylar, to which low-resistance connections cannot be easily made.

Acknowledgement:

It is a pleasure to thank Matt Partlow for his enthusiasm and feedback in doing this project.

References:

1. See, for example, W. H. Press, B. P. Flannery, S. A. Teukolsky, W. T. Vetterling, *Numerical Recipes: the Art of Scientific Computing*, Cambridge Univ. Press, 1986, especially Chapter 17: 'Partial Differential Equations', and section 17.5, 'Relaxation Methods for Boundary Value Problems'.

2. D. W. Koon and C. J. Knickerbocker, 'What do you measure when you measure resistivity', *Rev. Sci. Instrum.* **63**, 207 (1992).

BIOGRAPHY:

Daniel W. Koon is an Associate Professor of Physics at St. Lawrence University. He has studied the metal-insulator transition in arsenic-doped silicon by means of Hall measurements, and the effects of contact size and placement and of macroscopic impurities on measurement error in resistive and Hall measurements. Other research interests include fiber optic interferometry, and exploring alternatives to the traditional lecture-based class format. He is presently using an NSF Instrumentation and Lab Improvement grant (DUE-#9551787) to investigate optics as an alternate starting point in the traditional two-semester introductory physics sequence.