Resistive and Hall weighting functions in three dimensions

D. W. Koon
Department of Physics, St. Lawrence University, Canton, New York 13617

C. J. Knickerbocker
Department of Mathematics, St. Lawrence University, Canton, New York 13617

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The authors extend their study of the effect of macroscopic impurities on resistive and Hall measurements to include objects of finite thickness. The effect of such impurities is calculated for a series of rectangular parallelepipeds with two current and two voltage contacts on the corners of one square face. The weighting functions display singularities near these contacts, but these are shown to vanish in the two-dimensional limit, in agreement with previous results. Finally, it is shown that while Hall measurements principally sample the plane of the electrodes, resistivity measurements sample more of the interior of an object of finite thickness. © 1998 American Institute of Physics.

I. INTRODUCTION

Charge transport inhomogeneities can be divided into three categories, based on the length scale over which the transport property—resistivity (p) or Hall angle (θ)—varies. Microscopic inhomogeneities are those for which the length scale over which the property of interest varies is much smaller than the mean scattering length. Mesoscopic inhomogeneities are those for which the length scale is larger, but are still much smaller than the dimensions of the object being studied. Macroscopic inhomogeneities are whatever is left. The first two of these regimes have been widely studied, but it is only recently that the last has received much attention.

The authors have previously developed a procedure for calculating the effect of macroscopic inhomogeneities on resistive and Hall measurements, and have applied this technique to square, cloverleaf, cross, and bar specimens of infinitesimal thickness, as well as for moveable four-point probe arrays. In this article we extend this work to objects of finite, nonzero thickness.

First, we define the resistive (Hall) weighting function—the contribution of the local resistivity (Hall angle) of each volume element of a material to the total measured macroscopic resistance (Hall coefficient). The resistive weighting function, f(x,y,z), and the Hall weighting function, g(x,y,z), can be defined by:

\[ \langle \rho \rangle = \int p(x,y,z) f(x,y,z) dx dy dz / \int f(x,y,z) dx dy dz, \]

\[ \langle \Theta_H \rangle = \int \Theta_H(x,y,z) g(x,y,z) dx dy dz / \int g(x,y,z) dx dy dz. \]

where \( \langle \rho \rangle \) and \( \langle \Theta_H \rangle \) are the values measured for the resistivity and Hall angle of the entire object, \( p(x,y,z) \) and \( \Theta_H(x,y,z) \) are the local values of the corresponding transport properties, and the integration is over the volume of the object. For appropriately normalized weighting functions, the denominators can be ignored.

One can calculate the weighting functions from the effect that a point perturbation of either transport property will have on the measured property of the entire object. Thus, the calculation of weighting functions simplifies to a boundary value problem for an object with nonuniform transport properties, satisfying the modified Poisson equation of

\[ \nabla^2 \Phi = \frac{\nabla \cdot \nabla \Phi}{\rho} = \frac{\Theta_H}{\rho} \cdot (\nabla \Phi \times \hat{k}) + \nabla \Theta_H \cdot (\nabla \Phi \times \hat{k}). \]

The effect of a point inhomogeneity in resistivity (Hall angle) can be shown to be equivalent to the effect of placing an electric dipole at that same point, proportional to and parallel (perpendicular) to the direction of the electric field at that point. Such a problem can be solved exactly, for some geometries, by means of Green's functions, but in general can be solved numerically.

II. METHOD

The measurement geometry we considered is shown in Fig. 1. It consists of a rectangular parallelepiped with four electrodes on one of its square \( (a=b) \) faces. For resistive (Hall) measurements, the two current probes are on adjacent (opposite) corners of that face, and voltage probes are on the

![Diagram of measurement geometry](image-url)
FIG. 2. Resistivity weighting function in the plane of the electrodes, for thickness, \( c \), equal to \( 1/100 \) of the sides of the square face \( (c = a/100) \).

FIG. 3. Resistivity weighting function in the plane of the electrodes, for thickness, \( c \), equal to \( 1/10 \) of the sides of the square face \( (c = a/10) \).

FIG. 4. Resistivity weighting function for a cubic specimen \( (c = a) \), for (a) the plane of the electrodes, and (b) the plane opposite the plane of the electrodes.

FIG. 5. Hall weighting function calculated for an infinitesimally thin specimen.

FIG. 6. Hall weighting function in the plane of the electrodes, for thickness, \( c \), equal to \( 1/10 \) of the sides of the square face \( (c = a/10) \).

removing two corners. Calculations were done using a Green’s function approach as the thickness, \( c \), of the specimen was varied from 0.01 to 1 times the length of the other two edges \( (c = 0.01a, 0.1a, a) \).

Using the Green’s function approach for a rectangular parallelepiped is equivalent to calculating the electric potential due to an infinite number of image charges along all three of the coordinate axes. Calculation thus requires correspondingly longer converging times than for calculations in two dimensions, especially for thin \( (c \ll a, b) \) parallelepipeds.

III. RESULTS

The resistive weighting function \( f(x, y, z) \) is displayed in Figs. 2–4 for specimens 0.01 (Fig. 2), 0.1 (Fig. 3), and 1 (Fig. 4) times as thick as they are wide \( (c = 0.01a, 0.1a, a) \). Figure 4 shows the weighting function both in the plane of the electrodes [Fig. 4(a)] and in the opposite plane [Fig. 4(b)]. As expected, the two-dimensional (2D) approximation of Ref. 3 agrees well for the thinnest specimens, with less agreement as the specimen gets thicker. With increasing specimen thickness, singularities develop near the voltage and current probes in the plane of the electrodes. These singularities can be eliminated by averaging two independent
FIG. 8. Dependence of resistive weighting function (a) and Hall weighting function (b) on the distance of the slice from the plane of the electrodes. Both quantities are averaged over entire horizontal slices of the specimen. The dotted line in (a) is for a cube \((a = b = c)\); the solid line is for a specimen ten times thicker than it is wide \((c = 10a = 10b)\).

van der Pauw" measurements,\(^5\) as is the case for infinitesimally thin specimens.\(^3\) These singularities are absent in the plane opposite the plane of the electrodes, but the magnitude of the weighting function in this plane is also much smaller.

The Hall weighting function approached the 2D results of Fig. 5 in the thin-specimen limit. Its deviation from this limit for larger values of thickness, \(c\), is shown in Figs. 6 and 7 for specimens 0.1 and 1 times as thick as they are wide \((c = 0.1a, a)\). Figure 7 shows both the weighting function in the plane of the electrodes [Fig. 7(a)] and the weighting function in the opposite plane [Fig. 7(b)]. The same effects noted for resistive weighting functions occur here as well: with increasing thickness, singularities develop near the electrodes in the plane of the electrodes, while the weighting function vanishes in the plane opposite the electrodes. The main difference between the singularities for Hall and resistive weighting functions is that for resistive measurements, these singularities are both positive and negative, while for Hall measurements, they are purely positive. As a result, Hall singularities cannot be eliminated by averaging independent measurements.

The vanishing of the weighting functions in the plane opposite the electrodes can be seen in Fig. 8, which displays the resistivity [Fig. 8(a)] and Hall [Fig. 8(b)] weighting functions averaged over slices of the specimen parallel to the plane of the electrodes. Figure 8(a) shows the resistive weighting function for both a cube (dotted line) and an object ten times as thick as it is wide (solid line). The plane most heavily weighted in a resistance measurement lies about one-quarter of the width of the specimen below the electrodes, for sufficiently thick specimens. The planes furthest from the electrodes contribute only minimally to the measurement process, becoming especially irrelevant for specimens thicker than they are wide \((c > a, b)\). For Hall measurement, the plane most heavily weighted is the plane of the electrodes themselves.

IV. SUMMARY

In summary, we find that to minimize the effects of inhomogeneity in transport measurement, it is necessary to measure an object with a thickness much less than the distance between its electrodes. Furthermore, we find that resistive measurements sample more of the interior volume of the object under study, while Hall measurements primarily sample those layers of the object closest to the plane of the electrodes.