WEEK 6: Go to class 11, 12

Homework assignments

CLASS 11:

SECTION 3.3: SEPARATION OF VARIABLES

Skip Section 3.3.1. We will not do this in class. In class we will focus on the spherically symmetric case. (Section 3.3.2) Still, you should be aware of the existence of the rectangular case, and be able to look it up if you ever need to use it yourself.

SECTION 3.3.2: SPHERICALLY SYMMETRIC CASE AND LEGENDRE POLYNOMIALS

Consider a boundary value problems in which the electric potential, *V*, is defined on the surface of some sphere. Furthermore, for the sake of simplicity, let's assume that the potential has no azimuthal dependence. (i.e. The potential depends only on the angle *θ*, not on *φ*.) What can we say about the electric potential in the rest of space?

Because this problem has spherical symmetry, we will consider the electric potential to be a function of *r* and *θ* only: V=V(*r*,*θ*). In spherical coordinates the Laplacian is given by,...well ... (this is where the inside front cover of your text is very very useful: open it up and take a look at all of the useful reference material in it)

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

The method of separation of variables begins by hoping that we can write the electric potential as the product of two functions: We want a function of *r* only and another function of *θ* only: *V(r,θ)*=*R(r)θ*(*θ*). That removes the third term from the equation above, and we only have to solve the *r*-equation (Set first term of righthand side to zero) and the *θ*-equation (Set the second term of the righthand side to zero). If we plug this form into Laplace's equation -- $\nabla^2 V = 0$ -- we get

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-l-1}\right)P_l\left(\cos\theta\right)$$

where $P_l(\cos\theta)$ is a polynomial in the family known as the Legendre polynomials. We'll come back to this in a second. In general, the electric potential is the sum of an infinite series, where only integer values of ℓ work.

LEGENDRE POLYNOMIALS

You have seen Legendre polynomials before, if you applied Schrödinger's Equation solution to the hydrogen atom in Modern Physics. Why do these polynomials come up again here in classical electrostatics? The reason is that both mathematical problems involve (a) the Laplacian operator and (b) spherically symmetric boundary values. Since these polynomials come up in a number of places in physics, we should talk a little bit about them. First of all, the first few Legendre polynomials are given by

$$P_0(x) = 1 \quad P_2(x) = \tfrac{1}{2}\left(3x^2 - 1\right)$$
$$P_1(x) = x \quad P_3(x) = \tfrac{1}{2}\left(5x^3 - 3x\right)$$

Second, it would be useful to know how to derive the rest of the terms in this expansion. There are at least two useful ways of doing this. The first of these is known as the Rodrigues formula, after Benjamin Olinde Rodrigues. The Rodrigues formula for the Legendre polynomial P_l is

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l\left(x^2 - 1\right)^l$$

This equation is a bit of a bother. Why? Because if we want to calculate P_4, it doesn't help that we already know P_0 through P_3: we have to start from scratch again. A more convenient equation (one which the book doesn't provide you with) is the Iterative Formula

$$lP_l(x) = (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x)$$

We'll do some examples either on the board or as "daily" problems to give you some practice with this one.

Now how do we actually use these polynomials? Let's assume that we are given the electric potential on the sphere of radius R: $V(R,\theta)$ is given. The first task is to write this function in terms of the appropriate Legendre polynomials:

$$V(R,\theta) = \sum_{l=0}^{\infty} C_l P_l(\cos\theta)$$

Then we match this function to the form of the general solution to the spherically symmetric boundary value problem:

$$C_l = \begin{cases} A_l R^l & inside \\ B_l R^{-l-1} & outside \end{cases}$$

That's it. For really nasty problems, we would have to calculate an infinite number of coefficients in the expansion, very much like calculating the coefficients in a Fourier expansion. But, since I'm such a nice guy, we will only look at boundary value problems which include a handful of finite terms. (By the way, check out 1, 2, or 3, (These java apps may not work on all browsers.) or search the web using "fourier series java applet" to find some cool Fourier series simulations.)

WORKED PROBLEM:

Okay, here's a quick example of the kind of problem I might ask. Imagine that we are given a boundary condition that the electric potential on the surface of a sphere of radius R is equal to D $\cos^2\theta$, where D is some constant. We know that the solution should look like $V(r,\theta)\sum_{i=0}^{\infty}\left(A_l r^l + B_l r^{-l-1}\right)p_l(\cos\theta)$, which means that it should equal some linear combination of Legendre polynomials: $EP_2(\cos\theta) + FP_0(\cos\theta)$ +.... The trick is to figure out what E, F, etc. are. Now, for this particular problem, with $V = A\cos^2\theta$, the expansion will have no Legendre components higher than 2, since that is the highest-order term in *cosθ*.

After a little algebra, you should be able to convince yourself that $\cos^2\theta = [2P_2(\cos\theta) + P_0(\cos\theta)]/3$. From this we conclude that

$E = 2/3 = \left(A_2 R^2 + B_2 R^{-3}\right)$ and

$F = 1/3 = \left(A_0 R + B_0 R^{-1}\right)$.

If we are looking for the solution for $r>R$, then we set all the A_l's equal to zero: $B_2 = 2R^3/3$ and $B_0 = R/3$, or $V(r) = \frac{2R^3}{3r^3}P_1(\cos\theta) + \frac{R}{3r}P_0(\cos\theta)$.

If we are looking for the solution for $r < R$, then we set all the B_l's equal to zero: $A_2 = 2/3R^{-2}$ $A_2=(2/3)R^{-2}$ and $A_0 = 1/3$, or

$$V(r) = \frac{2r^2}{3R^3}P_1(\cos\theta) + \frac{1}{3}P_0(\cos\theta).$$

CLASS 12:

SECTION 3.4: MULTIPOLE EXPANSION AND DIPOLES

Now let's look at the the most general expression for the electric potential due to a charge distribution problem. It may look like this has nothing to do with all that Legendre polynomial stuff we have just done, but we shall see that it *does*.

Consider some distribution of charge located at the origin. The formal expression for the electric potential is the following integral:

$$V(\vec{r}) = \int \frac{k\rho(\vec{r}\,')d\tau'}{\mathscr{r}} = \int \frac{k\rho(\vec{r}\,')d\tau'}{|\vec{r}\text{-}r'|}$$

This is of course a very ugly integral to have to solve on one's own. First of all, the curly $\mathscr{r}$ in the denominator is the difference between the the position vector $\vec{r}$ at the point where you want to know the potential and the distance vector $\vec{r}\,'$ which is the location of the increment of charge, $dq' = \rho(\vec{r}\,')d\tau'$ you are integrating over. So not only is $\vec{r}\,'$ varying as you integrate over volume, but so is curly $\mathscr{r}$. The book executes a couple of tricks involving the Law of Cosines which allows us to rewrite this expression as an infinite power series which converges as we move further and further away from the charge distribution. What we get turns out to be

$$V(\vec{r}) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\theta')\rho(\vec{r}\,')d\tau'$$

There are those pesky Legendre polynomials again (with all the B_ℓ coefficients set to zero)! We can look at this expression term by term and write it as

$$V(\vec{r}) = \frac{K_0}{r} + \frac{K_1}{r^2} + \frac{K_2}{r^3} + \cdots$$

The first term has the exact same *r*-dependence that we would expect for the electric potential due to a single point charge. In fact, it turns out to equal *k* times the total charge of the charge distribution. We will call this the "monopole term". The next term falls off more rapidly. One can show that it is the electric potential that you would see if you have two equal but opposite electric charges located at the origin, but separated from each other by an infinitesimal distance. This is called the "dipole term". The terms following these two are known as the quadrupole term, the octupole term and so on. At large distances from the charge distribution, the monopoles term will swamp out all the other terms, *unless* the monopole term is zero, or very small. In that case, the dipole term will dominate, unless *it*, in turn, is zero.

What's the point here? That even uncharged objects can produce electrostatic fields. (In fact, cellophane does a good job of wrapping sandwiches because different parts of it attract via electrostatic forces, even though the entire piece is more or less uncharged.) In the next chapter, we shall look at a macroscopic phenomenon created by these dipole fields in uncharged matter. For that reason, it is worth looking a bit closer at electrostatic dipoles.

THE ELECTROSTATIC DIPOLE MOMENT

The second term of the expansion for the electric potential can be written as K_1/r^2, where

$$K_1 = k\hat{r}\cdot\vec{p}$$

where "r-cap" is a unit vector which points from the position of the dipole to the location in space where you want to know the electric potential. The charge distribution is equivalent, for our purposes, to two finite charges, +*Q* and -*Q*, separated by a distance *d*, and having a dipole moment, $\vec{p}$:

$$\vec{p} = q\vec{d}$$

where the vector points in the direction from the negative charge to the positive one.

THE ELECTRIC FIELD DUE TO A DIPOLE

Let's define the *z*-axis as the direction along which the dipole, $\vec{p}$, points, and let's say that the *x*-axis is in the plane perpendicular to *z*. Let's let $\vec{r}$ be the position vector pointing to the location where we want to calculate the electric field due to that dipole, and let θ be the angle between to *z*-axis and the $\vec{r}$ vector. We can define the components of the electric field in a couple of different ways. In spherical coordinates we have

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{kp}{r^3}\sin\theta \qquad E_r = -\frac{\partial V}{\partial r} = \frac{2kp}{r^3}\cos\theta$$

while in rectangular coordinates we have

$$E_x = -\frac{\partial V}{\partial x} = \frac{3kp}{r^3}\sin\theta\cos\theta \qquad E_z = -\frac{\partial V}{\partial z} = \frac{kp}{r^3}\left(3\cos^2\theta - 1\right)$$

This sounds straightforward enough, but once we start looking at the electric field at a point, say, midway between two dipoles, then it gets interesting because we need to define our *z* and $\vec{r}$ for *each* dipole, or *r* and θ relative to *each* dipole.

In the next chapter, we will look at the *effect* that an external electric field has on a dipole.