**WEEK 7: Go to class 13, 14**

Homework assignments

**CLASS 13:**

SECTION 3.4: MULTIPOLE EXPANSION AND DIPOLES

Now let's look at the the most general expression for the electric potential due to a charge distribution problem. It may look like this has nothing to do with all that Legendre polynomial stuff we have just done, but we shall see that it *does*.

Consider some distribution of charge located at the origin. The formal expression for the electric potential is the following integral:

$$V(\vec{r}) = \int \frac{k\rho(\vec{r}\,')d\tau'}{\mathscr{r}} = \int \frac{k\rho(\vec{r}\,')d\tau'}{|\vec{r}\text{-}r'|}$$

This is of course a very ugly integral to have to solve on one's own. First of all, the curly $\mathscr{r}$ in the denominator is the difference between the the position vector $\vec{r}$ at the point where you want to know the potential and the distance vector $\vec{r}\,'$ which is the location of the increment of charge, *dq'*= $\rho(\vec{r}\,')d\tau'$ you are integrating over. So not only is $\vec{r}\,'$ varying as you integrate over volume, but so is curly $\mathscr{r}$. The book executes a couple of tricks involving the Law of Cosines which allows us to rewrite this expression as an infinite power series which converges as we move further and further away from the charge distribution. What we get turns out to be

$$V(\vec{r}) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\theta')\rho(\vec{r}\,')d\tau'$$

There are those pesky Legendre polynomials again (with all the $B_\ell$ coefficients set to zero)! We can look at this expression term by term and write it as

$$V(\vec{r}) = \frac{K_0}{r} + \frac{K_1}{r^2} + \frac{K_2}{r^3} + \cdots$$

The first term has the exact same *r*-dependence that we would expect for the electric potential due to a single point charge. In fact, it turns out to equal *k* times the total charge of the charge distribution. We will call this the "monopole term". The next term falls off more rapidly. One can show that it is the electric potential that you would see if you have two equal but opposite electric charges located at the origin, but separated from each other by an infinitesimal distance. This is called the "dipole term". The terms following these two are known as the quadrupole term, the octupole term and so on. At large distances from the charge distribution, the monopoles term will swamp out all the other terms, *unless* the monopole term is zero, or very small. In that case, the dipole term will dominate, unless *it*, in turn, is zero.

What's the point here? That even uncharged objects can produce electrostatic fields. (In fact, cellophane does a good job of wrapping sandwiches because different parts of it attract via electrostatic forces, even though the entire piece is more or less uncharged.) In the next chapter, we shall look at a macroscopic phenomenon created by these dipole fields in uncharged matter. For that reason, it is worth looking a bit closer at electrostatic dipoles.

THE ELECTROSTATIC DIPOLE MOMENT

The second term of the expansion for the electric potential can be written as $K_1/r^2$, where

$$K_1 = k\hat{r} \cdot \vec{p}$$

where "r-cap" is a unit vector which points from the position of the dipole to the location in space where you want to know the electric potential. The charge distribution is equivalent, for our purposes, to two finite charges, +*Q* and -*Q*, separated by a distance *d*, and having a dipole moment, $\vec{p}$ :

$$\vec{p} = q\vec{d}$$

where the vector points in the direction from the negative charge to the positive one.

THE ELECTRIC FIELD DUE TO A DIPOLE

Let's define the *z*-axis as the direction along which the dipole, $\vec{p}$ , points, and let's say that the *x*-axis is in the plane perpendicular to *z*. Let's let $\vec{r}$ be the position vector pointing to the location where we want to calculate the electric field due to that dipole, and let *θ* be the angle between to *z*-axis and the $\vec{r}$ vector. We can define the components of the electric field in a couple of different ways. In spherical coordinates we have

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{kp}{r^3}\sin\theta \qquad E_r = -\frac{\partial V}{\partial r} = \frac{2kp}{r^3}\cos\theta$$

while in rectangular coordinates we have

$$E_x = -\frac{\partial V}{\partial x} = \frac{3kp}{r^3}\sin\theta\cos\theta \qquad E_z = -\frac{\partial V}{\partial z} = \frac{kp}{r^3}\left(3\cos^2\theta - 1\right)$$

This sounds straightforward enough, but once we start looking at the electric field at a point, say, midway between two dipoles, then it gets interesting because we need to define our *z* and $\vec{r}$ for *each* dipole, or *r* and *θ* relative to *each* dipole.

In the next chapter, we will look at the *effect* that an external electric field has on a dipole.

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**CLASS 14:**

CHAPTER 4: ELECTRIC FIELDS IN MATTER

I will take a completely different approach with this material then the text does. We will begin by looking at what effects electric fields can produce in macroscopic objects, and then we will consider the microscopic picture of how these effects arise.

DIELECTRIC CONSTANT

Consider a parallel plate capacitor. Often such a capacitor has some nonconducting material between the two plates. This material, called a dielectric, serves three principal purposes:

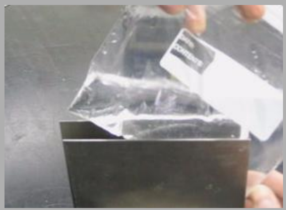
1. It separates the plates, keeping them from shorting each other out.
2. It prevents the two plates from sparking. Such sparking is known as dielectric breakdown, and occurs at about *3MV/m* in dry air and one-tenth that value in moist air.
3. The material increases the capacitance of the capacitor. This chapter is mostly about trying to understand why this happens.

We can start empirically (experimentally) by defining a quantity we call the dielectric constant, $\kappa$, or, as the book writes it, $\varepsilon_R$, the relative permittivity. In other words,

$$C = \kappa C_0 = \varepsilon_R C_0$$

where $C_0$ is the capacitance without the dielectric in place. (Notice that the dielectric constant is generally greater than one, just as the index of refraction is always greater than one.) Now, let's consider what happens when we put a dielectric between two parallel plates, from the standpoint of charge, voltage, and electric field. We will assume that we change none of the dimensions of the capacitor as we do this. First of all, since $C = Q/V$, increasing the capacitance without changing the charge means that we have a smaller voltage between the plates. Next, because $V = Ed$ for the uniform electric field between the plates, this means that putting the dielectric in place reduces the electric field between the two plates. Now, because the superposition of two electric fields adds like a vector, we can conclude that placing a dielectric between two capacitor plates has the effect of superimposing a somewhat smaller electric field, *E*, which points opposite the field produced by the charges on the plates. What is the origin of that extra field?

Inducing charge in a stream of [dielectric] water
http://www.youtube.com/watch?v=p1f6zLysilU



Dielectric constant of water
http://www.youtube.com/watch?v=-dzae5_BWus

THE DIELECTRIC SLAB

Let us picture a slab of dielectric material placed between two capacitor plates, filling nearly all the volume in between. Draw yourself a picture of this right now, a side view. Mark one of the plates as having positive charges on it, and the other plate as having negative charges. Now draw the electric field (outside the slab) pointing from the positively charged plate to the negatively charged plate. Next, draw the internal, reversed electric field inside the dielectric pointing in the opposite direction. Finally, label the edge of this slab from which this arrow originates as having a positive charge, and the other side with negative charge. What you should find at is that the side of the slab closest to the positively charged capacitor plate is loaded with negative charge, and the side closest to the negatively charged plate is loaded with positive charge.

What is the dielectric? Well, one thing it is *not* is a conductor. (Otherwise, E would be zero inside.)This means that these excess charges on the two faces of the slab do not come about from free charges flowing through the slab. Rather, they come about from charge separation *within* the atoms and molecules which make up the dielectric. We will sketch in class how a collection of fixed atoms can give rise to such a macroscopic separation of charge.

## POLARIZATION

In order for this macroscopic separation of charge to happen, we do need to have microscopic charge separation within the atom or molecule. The process by which the separation occurs on a microscopic level is called polarization. "Polarization" is also the name of a specific physical quantity, the average dipole moment per volume, which is given as

$$\vec{P} = \vec{p}\,/\,volume$$

There are two ways by which you can get polarization. Some molecules have a permanent dipole moment: water, for example. The electrons in the hydrogen atoms of water spend more time near the oxygen atom then they do at home with the hydrogen atoms, so this produces a permanent charge separation. On the other hand, in symmetric molecules such as $N_2$, $O_2$, $H_2$, there is no permanent dipole moment, but a temporary dipole moment can be induced. In the presence of an electric field, the electric field distorts the electron cloud surrounding the molecule in such a way that the average position of the electron is separated from the average position of the nuclei. The greater the electric field, the greater this separation. For sufficiently small electric fields, the average dipole moment is usually linear in the electric field for both permanent and induced dipole moments:

$$\vec{p} = \alpha \vec{E}_{app}$$

where $\alpha$ is the atomic polarizability. In Example 4.1, one sees that the atomic polarizability is of the order of $\varepsilon_0$ times the physical volume of the individual atom. This is approximately true throughout periodic table.

As for permanent dipoles in the presence of an electric field, random thermal collisions tend to counteract the effect of the electric field to align this permanent dipole, and so the average dipole moment increases with electric field. This means that we can use the last equation ( $\vec{p} = \alpha \vec{E}_{app}$) for permanent dipoles too.

## PERMANENT DIPOLES

In addition to the conceptual model above that shows how a dielectric can increase the capacitance of a pair of plates, we will also use this chapter as an opportunity to look at electric dipoles in general. There are a number of reasons for this. First of all, in materials that have no net electric charge, dipoles play a very important role, as we've just seen. The second reason is that for magnetic fields, there is no analogue to the electric charge. All magnetic fields are produced by dipoles. As it happens, the mathematics for *magnetic* dipoles is exactly the same as for *electric* dipoles, so maybe this is a good place to get our feet wet.

We can look at dipoles in electric fields from either the force approach or the energy approach. The net force on a dipole in a uniform electric field is zero, since the forces on the positive and negative ends of the dipole are identical. If the field is nonuniform, on the other hand, there will be a net nonzero force, but we will get to this later. What a dipole will experience in a uniform electric field is a torque. This is because, although the forces on the two ends of the dipole are equal in magnitude, they point in different directions, and thus act to twist the dipole. The torque on a dipole is given by

$$\vec{N} = \vec{p} \times \vec{E}$$

In the energy approach, we can calculate the potential energy of a dipole sitting in electric field as being equal to

$$U = -\vec{p} \cdot \vec{E}$$

Now, for the force in a nonuniform electric field. Since the force is the negative gradient of the potential energy --- $\vec{F} = -\vec{\nabla} U$ --- the force on a dipole in a non uniform field is

$$\vec{F} = \vec{\nabla}\left(\vec{p} \cdot \vec{E}\right)$$

(Try to wrap your brain around *that* equation. Write it out in component form. What does it mean? How would you calculate it?) In real situations, this is a rather messy equation to try to apply. The important thing I want you to get from this is that dipoles have net forces on them only in nonuniform fields.

Digression time. We don't see a lot of electric dipoles, so the story I'm going to tell here is the story of *magnetic* dipoles. If you have two refrigerator magnets, and you bring them close together, obviously they attract each other. The only way this can happen is if one of them produces a magnetic field which is nonuniform. If we put a collection of refrigerator magnets in a room that was filled with a large, but uniform magnetic field, all that would happen is that the magnets might twist to align themselves with the field, but they would not go flying through space. It is only because individual magnets produce wildly nonuniform magnetic fields that we get this effect that we are so familiar with, namely magnetic dipoles exerting net nonzero forces on each other.

SECTION 2

Skip most everything from Section 2 on, except for Example 4.6, Problems 4.18-4.21, and Section 4.4.3.

ENERGY IN ELECTRIC FIELDS IN MATTER

We saw in an earlier chapter that we can associate an energy density with electric field: $u = \frac{1}{2}\varepsilon_0 E^2$. What is the energy inside a dielectric between two parallel plate capacitor plates? Do we use the field outside the dielectric slab, or do we use the field inside? The answer, which will not derive, is a bit of a compromise:

$$u = \tfrac{1}{2}\varepsilon E^2 = \tfrac{1}{2}\kappa\varepsilon_0 E^2 = \kappa u \,,$$

where $u_0$ is the energy density without the dielectric. Now, since $\kappa$ is greater than one, this means that more energy is stored in the field when we add the dielectric, even though the electric field inside the dielectric is smaller than it was in that space before we inserted it. Why is this? Well, it takes a lot of energy to separate all those charges, or if the dielectric consists of permanent dipoles, to maintain the dipoles aligned in the face of all the thermal agitation which would tend to randomize their orientations. This also means that it requires energy to insert a dielectric into a capacitor.

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