WEEK 6: Go to class 11, 12

Homework assignments

CLASS 11:

SECTION 3.3: SEPARATION OF VARIABLES

Skip Section 3.3.1. We will not do this in class. In class we will focus on the spherically symmetric case. (Section 3.3.2) Still, you should be aware of the existence of the rectangular case, and be able to look it up if you ever need to use it yourself.

SECTION 3.3.2: SPHERICALLY SYMMETRIC CASE AND LEGENDRE POLYNOMIALS

Consider a boundary value problems in which the electric potential, *V*, is defined on the surface of some sphere. Furthermore, for the sake of simplicity, let's assume that the potential has no azimuthal dependence. (i.e. The potential depends only on the angle θ, not on φ.) What can we say about the electric potential in the rest of space?

Because this problem has spherical symmetry, we will consider the electric potential to be a function of r and θ only: $V=V(r,\theta)$. In spherical coordinates the Laplacian is given by,...well ... (this is where the inside front cover of your text is very very useful: open it up and take a look at all of the useful reference material in it)

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

The method of separation of variables begins by hoping that we can write the electric potential as the product of two functions: We want a function of r only and another function of θ only: $V(r,\theta)=R(r)\theta(\theta)$. That removes the third term from the equation above, and we only have to solve the r-equation (Set first term of righthand side to zero) and the θ-equation (Set the second term of the righthand side to zero). If we plug this form into Laplace's equation -- $\nabla^2 V = 0$ -- we get

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-l-1}\right)P_l(\cos\theta)$$

where $P_l(\cos\theta)$ is a polynomial in the family known as the Legendre polynomials. We'll come back to this in a second. In general, the electric potential is the sum of an infinite series, where only integer values of ℓ work.

LEGENDRE POLYNOMIALS

You have seen Legendre polynomials before, if you applied Schrödinger's Equation solution to the hydrogen atom in Modern Physics. Why do these polynomials come up again here in classical electrostatics? The reason is that both mathematical problems involve (a) the Laplacian operator and (b) spherically symmetric boundary values. Since these polynomials come up in a number of places in physics, we should talk a little bit about them. First of all, the first few Legendre polynomials are given by

$$P_0(x)=1 \quad P_2(x)=\tfrac{1}{2}\left(3x^2-1\right)$$
$$P_1(x)=x \quad P_3(x)=\tfrac{1}{2}\left(5x^3-3x\right)$$

Second, it would be useful to know how to derive the rest of the terms in this expansion. There are at least two useful ways of doing this. The first of these is known as the Rodrigues formula, after Benjamin Olinde Rodrigues. The Rodrigues formula for the Legendre polynomial P_l is

$$P_l(x)=\frac{1}{2^l\, l!}\left(\frac{d}{dx}\right)^l\left(x^2-1\right)^l$$

This equation is a bit of a bother. Why? Because if we want to calculate P_4, it doesn't help that we already know P_0 through P_3: we have to start from scratch again. A more convenient equation (one which the book doesn't provide you with) is the Iterative Formula

$$lP_l(x) = (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x)$$

We'll do some examples either on the board or as "daily" problems to give you some practice with this one.

Now how do we actually use these polynomials? Let's assume that we are given the electric potential on the sphere of radius R: $V(R,\theta)$ is given. The first task is to write this function in terms of the appropriate Legendre polynomials:

$$V(R,\theta) = \sum_{l=0}^{\infty} C_l P_l(\cos\theta)$$

Then we match this function to the form of the general solution to the spherically symmetric boundary value problem:

$$C_l = \begin{cases} A_l R^l & inside \\ B_l R^{-l-1} & outside \end{cases}$$

That's it. For really nasty problems, we would have to calculate an infinite number of coefficients in the expansion, very much like calculating the coefficients in a Fourier expansion. But, since I'm such a nice guy, we will only look at boundary value problems which include a handful of finite terms. (By the way, check out 1, 2, or 3, (These java apps may not work on all browsers.) or search the web using "fourier series java applet" to find some cool Fourier series simulations.)

WORKED PROBLEM:
Okay, here's a quick example of the kind of problem I might ask. Imagine that we are given a boundary condition that the electric potential on the surface of a sphere of radius R is equal to $D \cos^2\theta$, where D is some constant. We know that the solution should look like $V(r,\theta)\sum_{i=0}^{\infty}\left(A_l r^l + B_l r^{-l-1}\right)p_l(\cos\theta)$, which means that it should equal some linear combination of Legendre polynomials: $EP_2(\cos\theta) + FP_0(\cos\theta)$ +.... The trick is to figure out what E, F, etc. are. Now, for this particular problem, with $V = A\cos^2\theta$, the expansion will have no Legendre components higher than 2, since that is the highest-order term in $cos\theta$.

After a little algebra, you should be able to convince yourself that $\cos^2\theta = \left[2P_2(\cos\theta) + P_0(\cos\theta)\right]/3$. From this we conclude that

$$E = 2/3 = \left(A_2R^2 + B_2R^{-3}\right) \quad \text{and}$$
$$F = 1/3 = \left(A_0R + B_0R^{-1}\right).$$

If we are looking for the solution for $r>R$, then we set all the A_l's equal to zero: $B_2 = 2R^3/3$ and $B_0 = R/3$, or $V(r) = \frac{2R^3}{3r^3}P_1(\cos\theta) + \frac{R}{3r}P_0(\cos\theta)$.

If we are looking for the solution for $r < R$, then we set all the B_l's equal to zero: $A_2 = 2/3R^{-2}$ $A_2 = (2/3)R^{-2}$ and $A_0 = 1/3$, or

$$V(r) = \frac{2r^2}{3R^3}P_1(\cos\theta) + \frac{1}{3}P_0(\cos\theta).$$

CLASS 12:
EXAM II (takehome)