WEEK 5: Go to class 9, 10

Homework assignments

CLASS 9:

ENERGY STORED IN A CAPACITOR, *Still* Section 2.5.4

It is easy to calculate the energy density between two parallel plates of charge and multiply by the volume between the plates to get

$$U = \tfrac{1}{2}QV = \tfrac{1}{2}CV^2 = \tfrac{1}{2}Q^2/C$$

It turns out that this result is true for *all* capacitors, so knowing the capacitance of a pair of conductors can save us a lot of time when it comes to calculating how much energy is stored inside the electric field between them. (Otherwise, we would have to calculate the energy density, *u*, throughout space, and then *integrate that*.) In case you don't believe a capacitor can store energy, check out this video...

Capacitor stores energy
http://www.youtube.com/watch?v=-3IbAerYj8I

By the way, now we can calculate the amount of energy dissipated when that spark flies between your finger and the doorknob. Assume a 1cm spark, as in last Thursday's notes, and 5cm for the radius of your finger. Since $V=30kV$ and $C=0.55pF$, the total energy is $\tfrac{1}{2}CV^2 = 0.25mJ$, not very large, but enough to get your attention. Obviously, the further the spark flies, the larger the amount of energy released.

SECTION 3.1: LAPLACE'S EQUATION

Laplace's Equation, $\nabla^2 V = 0$, has a very peculiar, but useful property. Here it is, as it manifests itself in 1D, 2D, or 3D problems. We'll focus on 2D problems in this course.

1D:	solution $V(x) = mx + b$	$V(x) = [V(x-a) + V(x+a)]/2$	No local maxima or minima
2D:		$V(x,y) = average\ on\ circle$	No local maxima or minima
3D:		$V(x,y,z) = average\ on\ sphere$	No local maxima or minima

We'll show mathematically why this is, later.

BOUNDARY CONDITIONS, Section 3.1.5

There are two classes of boundary conditions that commonly exist, as well as hybrid combinations of the two. Physicists commonly refer to these two classes by name. If you need to learn more about them after this course, it helps to recognize these names. Besides, I will need to use these names myself in a few minutes. Also, there is a wonderful theorem or two that says that if you have either one or both of these conditions holding along the entire boundary of some region of space, then there is exactly one solution to Laplace's equation inside those boundaries. In other words, once you've found *a* solution, you can quit: it is the *only* solution.

The two conditions are called "Dirichlet" conditions, wherein the electric potential, *V*, is specified on the boundary, and "Neumann" conditions, in which the derivative of the potential *normal* to the surface (The $\partial V/\partial n$ we mentioned last week) is given. An example of Dirichlet conditions would be if the four walls of the lecture room were held at four separate "voltages". A Neumann example would be if a certain constant current were made to flow between two opposing walls, with perhaps the other walls, floor, and ceiling electrically insulated and passing no current. A combination of these conditions is also possible, with some walls “Dirichlet” and others “Neumann”.

METHOD OF RELAXATION (not in book: See these notes and the professor's handout instead, based on a student's project in this class years ago.)
As we saw last class, the electric potential in 2D or 3D, assuming Laplace's equation, is equal to the average of the potential at points equidistant from the point of interest, that is, the points on a circle in 2D, or on a sphere in 3D. This allows us to use a very powerful approximation technique. Let's consider 2D, since it is the dimensionality of your notepad, the blackboard, and a computer screen.

Consider a boundary-value problem defined in two dimensions -- a square, for example. We can replace the continuous values of *x* and *y* coordinates inside the square with gridpoints, as long as we make the points close enough. Laplace's equation then tells us that the electric potential at a point on our grid will equal the average of the values of potential at neighboring points. For the grid I've displayed below, that means that $V_{1,1} = \left(V_{0,1} + V_{2,1} + V_{1,0} + V_{1,2}\right)/4$.

$$\begin{matrix} V_{0,0} & V_{0,1} & V_{0,2} \\ V_{1,0} & V_{1,1} & V_{1,2} \\ V_{2,0} & V_{2,1} & V_{2,2} \end{matrix}$$

To apply this technique, create a spreadsheet in Excel or create a matrix in some programming language. Place your initial guess for the potential (the initial "guess" can be all zeros, although that’s not a very creative guess.) in the matrix/spreadsheet, then apply the formula above to re-evaluate the potential in every nook and cranny of the matrix/spreadsheet. Eventually, your values of $V_{i,j}$ will converge or "relax" to values approximating the function you were trying to solve for. This is called the relaxation technique. Here is somebody setting up an Excel spreadsheet to do this.

Using Excel to calculate potential
http://www.youtube.com/watch?v=g6wW1f11uwM

It’s hard to follow the details of what this person is doing, unless you can read what looks like Korean or Chinese. You will be doing a HW problem and a Daily where you will solve a boundary value problem [BVP] using Excel.

Simple, no? I have to add a couple more details before you can actually get it to fly right. First, you need to apply boundary conditions, or else your $V_{i,j}$=0 initial guess will be your final answer as well. What if $V_{1,0}$ is a boundary point, and what if the boundary condition at that point is that the normal component of the E-field is zero (a Neumann boundary condidition)? Imagine a phantom gridpoint, $V_{1,-1}$, to the left of our nine gridpoints. The boundary condition I've just imposed is equivalent to saying that $V_{1,-1}=V_{1,1}$. (Think about why that is.) Consequently, averaging the four neighbors, as we did above becomes:

$$V_{1,0} = \left(V_{0,0} + V_{2,0} + V_{1,-1} + V_{1,1}\right)/4 = \left(V_{0,0} + V_{2,0} + 2V_{1,1}\right)/4 .$$

In other words, on the boundary cell you still average the neighbors, but you count the neighbor interior to the boundary cell twice.

There are ways of speeding up the convergence of the relaxation technique. One is to recruit "neighbors" from further afield. In other words, instead of just averaging the nearest neighbors, you include second-nearest neighbors, third-nearest neighbors, etc. There are various schemes for figuring out how to weight each set of neighbors in the averaging, but this is clearly beyond the scope of this course, and besides, this is not nearly as elegant as our much simpler approach.

A second, much-easier-to-program approach is called "over-relaxation". It makes use of the fact that relaxation converges monotonically in a single direction. In other words, successive values for $V_{1,1}$ might equal 1.000V, 1.100V, 1.110V, 1.111V. Why not anticipate the direction of change in V? One way to do this is to define a quantity *α*, usually between 1 and 2, and changing our original equation

from $$V_{1,1} = \left(V_{0,1} + V_{2,1} + V_{1,0} + V_{1,2}\right)/4$$
to $$V_{1,1} = (1-\alpha)V_{1,1} + \alpha\left(V_{0,1} + V_{2,1} + V_{1,0} + V_{1,2}\right)/4$$

where one plugs the present values of the potentials $V_{i,j}$ on the right hand side to calculate the new value of $V_{1,1}$ for the left hand side.

For what it's worth, what we are doing here by discretizing two-D space is similar to the painting style known as pointillism. Expect the instructor to bring some examples to class (Pointillism is of course the precursor to digital photography, but I digress.) and enjoy this pointillist masterpiece:

Mythbusters create the 80msec Mona Lisa
http://www.youtube.com/watch?v=fKK933KK6Gg

CLASS 10:

SECTION 3.2: METHOD OF IMAGES

There is a class of problems for which we can use a shortcut called the Method of Images. The downside of the shortcut is that it works for only a very small minority of all boundary-value problems. On the other hand, it is a great shortcut for the problems for which it *does* work. The other techniques we study in this chapter will be general techniques, but not this one: it works for only *certain* problems.

Consider a point charge near an infinite flat conductor. From the properties of conductors that we studied before, we know that the component of the electric field parallel to the surface of the conductor is zero. Now imagine that we remove the conductor and place a charge of opposite sign from the one we are considering at any point directly opposite the conductor from it. (In other words, if this conductor had been a mirror, put this "image charge" where the optical image of the charge of interest would be.) It turns out that for this system of charges, the electric field at the plane where the conductor used to be will also have zero electric field component parallel to the surface.

There is an important Uniqueness Theorem for boundary-value problems that states that, if the electric potential or the "normal derivative" of a possible solution to an electrostatics problem matches the conditions at the boundary of the problem, then that solution will be valid throughout the volume of the problem, and, in fact, will be the unique solution to the problem. This saves as a ton of work, because it allows us to substitute this system with the "image charge" for the original problem.

We have already used the term "image", which reminds us of (optical) mirrors. How much further can we stretch the optical analogy? The charge and its image charge are of opposite polarity. Compare this to yourself and your "image" self, as you look in the mirror. If you're wearing a sweatshirt with writing on it, your "image" self will be wearing a sweatshirt in which the writing is reversed. Now take two mirrors and hinge them so that they come together at a 90^o angle. You will now notice three images: one in front of you and one each on either side. The one in front of you will be wearing a sweatshirt with the writing easily readable; the two side images will have the writing reversed. If we compare this with the case in which we have two plane conductors that come together to form a 90^o angle, we see that there will be three "image charges" behind this conductor. They will be located at the same places where the optical images are formed, and the two on either side will be of opposite charge from the original charge and the charge directly in front will be the same sign.

In class, we will look at hinged mirrors, drawing the appropriate analogies for the method of images. One interesting question to ask is what happens when the two mirrors are hinged at an arbitrary angle? Are there certain angles for which this method works better than others? Think about this and bring your answer in with you to class. (Oh, and try to figure out how many animals are in each scene of this clip...)

Animal Mirror Fun:
http://www.youtube.com/watch?v=VxeNveyrT_o

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