# MATH 348: FINANCIAL MATHEMATICS
# NOTES ON BONDS

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Abstract. Bonds are best viewed as a combination of an annuity and a forward. We also give a proper proof of Proposition 2.5.

## 1. Valuing bonds

A *bond* is a promise of a sequence of future payments. We are treating bonds as riskless, meaning there is no chance of default. Default risk can be priced in. The bond payments are of two types, a regular *coupon* payment $C$, which we shall take as occurring once per time period (eg. year) and a *face value* amount $F$ paid at a fixed future time $T$, the *maturity date.* Since the future payments, that is, the income stream from the coupons and the face value are known, the key problem is determining the present value of the bond, given fixed, known interest rate $r$, or, alternatively, determining the implied interest rate given the present value. The stream of coupon payments amounts to an annuity and the face value payment at time $T$ is a forward contract.

Suppose a bond pays a coupon $C$ each year and has a face value of $F$ payable at time $T = n$ (years). Assume a fixed interest rate $r$.

### 1.1. Zero-Coupon Bonds.

The case $C = 0$ is called a *zero-coupon* bond. Here we are only concerned with the forward contract. From Chapter 1, we have that the value of a forward contract in $n$ years at interest rate $r$ is

$$F = V(0)(1+r)^n. \tag{1.1}$$

Solving for the present value, we get

$$V(0) = \frac{F}{(1+r)^n}.$$

Note that the present price of a bond is inversely related to the interest rate, but not in a linear fashion.

If we want to know the value at some future time $t$, $0 \le t \le n$, then there will be time $(n-t)$ remaining for the bond and we have

$$V(t) = \frac{F}{(1+r)^{(n-t)}}.$$

Alternatively, given the present value $V(0)$ and the face value, $F$, one can solve equation 1.1 to find the implied interest rate

$$r = \left(\frac{F}{V(0)}\right)^{\frac{1}{n}} - 1.$$

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*Example* 1. Suppose a zero-coupon bond has a face-value $F = 100$ and matures in two years so that $n = 2$, and that the interest rate is $r = 0.08$, or $r = 8\%$. Then the present value of the bond is

$$V(0) = \frac{F}{(1+r)^n} = \frac{100}{(1+.08)^2} \cong 85.73.$$

The value at time $t = 1$ is

$$V(1) = \frac{F}{(1+r)^(n-t)} = \frac{100}{(1+.08)^{(2-1)}} \cong 92.59.$$

*Example* 2. Conversely, suppose a zero-coupon bond with a three-year maturity and face-value of $F = 100$ currently trades at $V(0) = 85$, then the implied interest rate is

$$r = \left(\frac{F}{V(0)}\right)^{\frac{1}{n}} - 1 = \left(\frac{100}{85}\right)^{\frac{1}{3}} - 1 \cong 0.056,$$

or approximately 5.6%.

1.2. **Coupon Bonds.** Now suppose $C \neq 0$. $C$ is usually considered positive, although there is no real need for this in the theory. Each future coupon payment has to be discounted to determine the present value. Since the sequence of payments is fixed and regular, the coupon payments form an annuity and from Chapter 1, we have that the present value of an annuity of regular payment $C$ for $n$ years with fixed interest rate $r$ is

$$V(0) = C \times PA(r, n),$$

where

$$PA(r, n) = \frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right).$$

Putting together the coupon payments and the forward face-value contract, we have

$$\begin{aligned} V(0) &= C \times PA(r,n) + \frac{F}{(1+r)^n} \\ &= C\frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right) + \frac{F}{(1+r)^n}. \end{aligned} \tag{1.2}$$

*Example* 3. A 10-year bond has a face-value of $F = 100$ and coupon $C = 6$. If the interest rate is $r = 0.1$, then the current price of the bond is

$$\begin{aligned} V(0) &= C\frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right) + \frac{F}{(1+r)^n} \\ &= 6\frac{1}{0.1}\left(1 - \frac{1}{(1+0.1)^{10}}\right) + \frac{100}{(1+0.1)^{10}} \\ &\cong 75.42 \end{aligned}$$

1.3. **Par-value.** Since the coupon $C$ and the face-value $F$ are fixed, we can write the relation between them as $C = iF$. Then $i$ is called the *coupon rate*. Substituting into Equation 1.2, we have

$$\begin{aligned} V(0) &= iF \times PA(r,n) + \frac{F}{(1+r)^n} \\ &= F\frac{i}{r}\left(1 - \frac{1}{(1+r)^n}\right) + \frac{F}{(1+r)^n}. \end{aligned} \tag{1.3}$$

This formula suggests a relationship between $V(0)$, $F$, $i$ and $r$ that is given in the book as Proposition 2.5.

**Proposition 4.** *With the notation as above, $V(0) = F$ if and only if $i = r$.*

*Proof.* From Equation 1.3, we have

$$\frac{V(0)}{F} = \frac{i}{r}\left(1 - \frac{1}{(1+r)^n}\right) + \frac{1}{(1+r)^n}$$

Suppose $i = r$. Then $\frac{i}{r} = 1$ and the right-hand side simplifies, so that $\frac{V(0)}{F} = 1$, or $V(0) = F$.

Conversely, if $V(0) = F$, then we have

$$1 = \frac{i}{r}\left(1 - \frac{1}{(1+r)^n}\right) + \frac{1}{(1+r)^n}.$$

Then,

$$1 - \frac{1}{(1+r)^n} = \frac{i}{r}\left(1 - \frac{1}{(1+r)^n}\right),$$

so $\frac{i}{r} = 1$. □

If $V(0) = F$, the bond is selling in the present at its face-value and is said to be trading at *par*. Although not so obvious from the formulas, if $i < r$, then $V(0) < F$ (think zero-coupon bonds where $i = 0$) and the bond is trading *below* par. If $i > r$, then $V(0) > F$ and the bond is trading *above* par. The No-Arbitrage Principle makes these relationships clear. Borrow at interest rate $r$ to invest in bond returning coupon rate $i$. If $i = r$, the coupons exactly cover the loan interest so the maturity value should equal the original loan. If the coupons pay less than the interest ($i < r$), then the future value must be higher to compensate ($V(0) < F$). If the coupons pay more than the interest ($i > r$), then the maturity value should be less than the original loan ($V(0) > F$).